# Pazy's fixed point theorem with respect to the partial order in uniformly convex Banach spaces

Yisheng Song[*]  Rudong Chen[†]

**Abstract**

In this paper, the Pazy's Fixed Point Theorems of monotone $\alpha-$nonexpansive mapping $T$ are proved in a uniformly convex Banach space $E$ with the partial order "$\leq$". That is, we obtain that the fixed point set of $T$ with respect to the partial order "$\leq$" is nonempty whenever the Picard iteration $\{T^n x_0\}$ is bounded for some initial point $x_0$ with $x_0 \leq T x_0$ or $T x_0 \leq x_0$. When restricting the demain of $T$ to the cone $P$, a monotone $\alpha-$nonexpansive mapping $T$ has at least a fixed point if and only if the Picard iteration $\{T^n 0\}$ is bounbed. Furthermore, with the help of the properties of the normal cone $P$, the weakly and strongly convergent theorems of the Picard iteration $\{T^n x_0\}$ are showed for finding a fixed point of $T$ with respect to the partial order "$\leq$" in uniformly convex ordered Banach space.

**Key Words and Phrases:** Ordered Banach space, fixed point, monotone $\alpha$-nonexpansive mapping, convergence.

**2000 AMS Subject Classification:** 47H06, 47J05, 47J25, 47H10, 47H17, 49J40, 47H04, 65J15.

## 1 Introduction

In 1971, Pazy [20] proved the following fixed point theorem for a nonexpansive mapping in Hilbert spaces, which is referred to as **Pazy's Fixed Point Theorem**.

**Theorem P.** *Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a nonexpansive mapping. Then the following are equivalent:*

    (i) *$\{T^n x\}$ is a bounded sequence in $C$ for some $x \in C$.*

    (ii) *$T$ has at least a fixed point in $C$.*

Recently, many mathematical workers studied the Pazy's Fixed Point Theorem for many distinct types of nonlinear mappings. In 2008, Kohsaka and Takahashi [13] showed Pazy's Fixed Point Theorem of a nonspreading mapping in a Hilbert space $H$. A mapping $T : C \to C$ is called a *nonspreading mapping* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \text{ for all } x, y \in C.$$

They showed that $T$ has a fixed point whenever $\{T^n x\}$ is a bounded sequence for some $x \in C$. The same result was obtained by Takahashi [28] for a hybrid mapping in a Hilbert space $H$. A mapping $T : C \to C$ is called a *hybrid mapping* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle \text{ for all } x, y \in C.$$

In 2011, Takahashi and Yao [29] proved Pazy's Fixed Point Theorem of a $TJ$-mapping in a Hilbert space $H$. A mapping $T : C \to C$ is called a *$TJ$-mapping* if

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 \text{ for all } x, y \in C.$$

In 2012, Lin and Wang [14] showed Pazy's Fixed Point Theorem for an $(a, b)$-monotone mapping in a Hilbert space $H$. A mapping $T : C \to C$ is called an *$(a, b)$-monotone mapping* if for all $x, y \in C$,

$$\langle x - y, Tx - Ty \rangle \geq a\|Tx - Ty\|^2 + (1 - a)\|x - y\|^2 - b\|x - Tx\|^2 - b\|y - Ty\|^2,$$

where $a \in (\frac{1}{2}, \infty)$ and $b \in (-\infty, a)$ Very recently, Song et. al. [22] firstly studied Pazy's Fixed Point Theorem for a monotone nonexpansive mapping in a uniformly convex Banach space with the partial order "$\leq$". Let $T$ be a mapping with domain $D(T)$ and range $R(T)$ in an ordered Banach space $E$ endowed with the partial order "$\leq$". Then $T : D(T) \to R(T)$ is said to be *monotone nonexpansive* ([2]) if for all $x, y \in D(T)$ with $x \leq y$,

$$Tx \leq Ty \text{ and } \|Tx - Ty\| \leq \|x - y\|.$$

Clearly, a monotone nonexpansive mapping may be discontinuous. In 2015, Bachar and Khamsi [2] introduced the concept of a monotone nonexpansive mapping and studied common approximate fixed points of a monotone nonexpansive semigroup. Dehaish and Khamsi [4] proved some weak convergence theorems of the Mann iteration for finding some order fixed points of monotone nonexpansive mappings in uniformly convex ordered Banach spaces.

The *Mann iteration* was introduced by Mann [16] in 1953 for finding a fixed point of a nonexpansive mapping $T$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n \text{ for each positive integer } n.$$

There had be many convergence conclusions of such an iteration in the past several decades. For more details, see Liu [15], Narghirad et al. [17], Suzuki [26], Song [24], Opial [19], Kim et al. [12], Okeke and Kim [18], Berinde [3], George and Shaini [6], Gu and Lu [10], Zhou et al. [32], Song and Wang [23], zhang and Su [31], Zhou [33] and the reference therein. Very recently, Song et. al. [21] studied weak convergence of the Mann iteration for monotone $\alpha$-nonexpansive mappings in a uniformly convex Banach space with the partial order "$\leq$" under the coefficient condition "$\limsup\limits_{n\to\infty} \beta_n(1 - \beta_n) > 0$" or "$\liminf\limits_{n\to\infty} \beta_n(1 - \beta_n) > 0$", which excludes $\beta_n = 0$ or $\frac{1}{n}$.

If $\beta_n = 0$ in Mann iteration, then such an iteration is called *Picard iteration*. In this paper, we will study the convergence of Picard iteration in a uniformly convex Banach space with the partial order "$\leq$". More precisely, we consider the Pazy's Fixed Point Theorem for monotone $\alpha$-nonexpansive mappings in a uniformly convex Banach space with the partial order "$\leq$" in Section 3. That is, for a monotone $\alpha$-nonexpansive mapping $T$, the boundedness of the Picard iteration $\{T^n x_0\}$ for some $x_0$ with $x_0 \leq Tx_0$ or $x_0 \geq Tx_0$ implies that ordered fixed point set $F_{\geq}(T) \neq \emptyset$ or $F_{\leq}(T) \neq \emptyset$. Furthermore, for a monotone $\alpha$-nonexpansive mapping $T$ defined on a closed convex cone $P$ with respect to the partial order "$\leq$", its fixed point set $F(T) \neq \emptyset$ if and only if the Picard iteration $\{T^n 0\}$ is bounded (where 0 is the origin). In Section 4, under the frame of the partial order "$\leq$" of a Banach space $(E, \|\cdot\|)$ defined by the normal cone $P$, we will show the weak and strong convergence of Picard iteration of a monotone $\alpha$-nonexpansive mapping.

## 2 Preliminaries and basic results

Throughout this paper, let $E$ be an ordered Banach space with the norm $\|\cdot\|$ and the partial order "$\leq$". Let $F(T) = \{x \in E : Tx = x\}$ denote the set of all fixed points of a mapping $T$.

Let $E$ be a real Banach space and $P$ be a subset of $E$. $P$ is called a *closed convex cone* if $P$ is nonempty closed, and $P \neq \{0\}$ with $P \cap (-P) = \{0\}$, and $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers $a, b$. The *partial order* "$\leq$", "$<$" and "$\ll$" with respect to $P$

in $E$ are defined as follows: for all $x, y \in E$,

$$x \leq y \ \text{ if and on if } \ y - x \in P,$$
$$x < y \ \text{ if and on if } \ y - x \in P \text{ and } y \neq x,$$
$$x \ll y \ \text{ if and on if } \ y - x \in \mathring{P} \text{ whenever } \mathring{P} \neq \emptyset,$$

where $\mathring{P}$ is the interior of $P$. The partial order "$\geq$" is defined by

$$x \geq y \ \text{ if and on if } y \leq x.$$

Then we have

$$x = y \ \text{ if and on if } y \leq x \text{ and } y \geq x. \tag{2.1}$$

In the sequel, $[x, y]$ stands for the order interval. An *order interval* $[x, y]$ for all $x, y \in E$ is given by

$$[x, y] = \{z \in E : x \leq z \leq y\}. \tag{2.2}$$

Clearly, the order interval $[x, y]$ are closed and convex by the definition of the partial order "$\leq$", which implies that

$$x \leq tx + (1 - t)y \leq y \tag{2.3}$$

for all $t$ with $0 \leq t \leq 1$ and all $x, y \in E$ with $x \leq y$.

**Definition 2.1.** *([5]) Let $(E, \| \cdot \|)$ be an ordered Banach space with the partial order "$\leq$" with respect to a cone $P$.*

(1) *A sequence $\{x_n\} \subset E$ is called* monotonic *if $\{x_n\}$ is either increasing, i.e., $x_n \leq x_{n+1}$ for all positive integer $n$, or decreasing, i.e., $x_n \geq x_{n+1}$ for all positive integer $n$.*

(2) *A subset $C$ of $E$ is called* bounded above *if $C$ has an upper bound with respect to "$\leq$", i.e., there exists $y \in E$ such that $x \leq y$ for all $x \in C$, and the least upper bound of $C$ with respect to "$\leq$" is called the supremum of $C$, denote $\sup C$ if it exists.*

(3) *A subset $C$ of $E$ is called* bounded below *if $C$ has a lower bound with respect to "$\leq$", i.e., there exists $z \in E$ such that $z \leq x$ for all $x \in C$, and the greatest lower bound of $C$ with respect to "$\leq$" is called the infimum of $C$, denote $\inf C$ if it exists.*

(4) *The cone $P$ is called* normal *if there exists a constant $\gamma > 0$ such that $\|x\| \leq \gamma \|y\|$ for all $x, y \in E$ with $0 \leq x \leq y$ or equivalently, if the order interval $[x, y]$ for all $x, y \in E$ with $x \leq y$ is bounded with respect to the norm $\| \cdot \|$.*

(4) *The cone $P$ is called* minihedral *if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if each set which is bounded above has a supremum.*

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**Lemma 2.1.** *([25]) Let $(E, \| \cdot \|)$ be an ordered Banach space with the partial order "$\leq$" with respect to a cone $P$.*

(i) *Assume that $\{x_n\}$ and $\{y_n\}$ are two sequences on $E$ such that*

$$x_n \leq y_n \text{ for each positive integer } n.$$

*If $x_n$ and $y_n$ weakly converges to $x$ and $y$, respectively, then*

$$x \leq y.$$

(ii) *The cone $P$ is normal if and only if there exists a equivalent norm $\| \cdot \|_1$ of $E$ such that*

$$\|x\|_1 \leq \|y\|_1 \text{ for all } x, y \in E \text{ with } 0 \leq x \leq y.$$

**Definition 2.2.** *([2, 21]) Let $K$ be a nonempty closed convex subset of an ordered Banach space $(E, \leq)$. A mapping $T : K \to E$ is said to be:*

(1) *monotone ([2]) if $Tx \leq Ty$ for all $x, y \in K$ with $x \leq y$;*

(2) *monotone nonexpansive ([2]) if $T$ is monotone and*

$$\|Tx - Ty\| \leq \|x - y\|$$

*for all $x, y \in K$ with $x \leq y$.*

(3) *monotone $\alpha$-nonexpansive([21]) if $T$ is monotone and for some $\alpha < 1$,*

$$\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$$

*for all $x, y \in K$ with $x \leq y$;*

(4) *monotone quasi-nonexpansive([21]) if $F(T) \neq \emptyset$ and*

$$\|Tx - p\| \leq \|x - p\|$$

*for all $p \in F(T)$ and all $x \in K$ with $x \leq p$ or $x \geq p$.*

Obviously, a monotone nonexpansive mapping is monotone 0-nonexpansive. In the sequel, we will use the fixed point sets with the partial order $F_\leq(T)$ and $F_\geq(T)$ given by

$$F_\leq(T) = \{p \in F(T) : p \leq x \text{ for some } x \in K\}$$

and

$$F_\geq(T) = \{p \in F(T) : x \leq p \text{ for some } x \in K\}.$$

**Lemma 2.2.** *([21, Lemma 2.1]) Let $K$ be a nonempty closed convex subset of an ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping. Then*

*(1) $T$ is monotone quasi-nonexpansive if $F_\leq(T) \neq \emptyset$ or $F_\geq(T) \neq \emptyset$;*

*(2) for all $x, y \in K$ with $x \leq y$ (or $y \leq x$),*

$$\|Tx - Ty\|^2$$
$$\leq \|x - y\|^2 + \frac{2\alpha}{1 - \alpha}\|Tx - x\|^2 + \frac{2|\alpha|}{1 - \alpha}\|Tx - x\|(\|x - y\| + \|Tx - Ty\|).$$

**Definition 2.3.** *Let $E$ ba a Banach space. Then*

*(1) a function $\delta_E : [0, 2] \to [0, 1]$ is said to be the* modulus of convexity *of $E$ if*

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2}; \ \|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \varepsilon\};$$

*(2) a number $\varepsilon_0(E)$ is said to be the* characteristic of convexity *of $E$ if*

$$\varepsilon_0(E) = \sup\{\varepsilon \in [0, 2]; \delta_E(\varepsilon) = 0\}.$$

**Definition 2.4.** *A Banach space $E$ is said to be*

*(1)* uniformly convex *if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ or equivalently if $\varepsilon_0(E) = 0$;*

*(2)* strictly convex *if $\delta_E(2) = 1$.*

The following properties of the modulus of convexity of a Banach space $E$ were found in the references [1, 8, 9].

**Lemma 2.3.** *Let $E$ be a Banach space with the modulus of convexity $\delta_E(\cdot)$ and the characteristic of convexity $\varepsilon_0$. Then we have the following:*

*(i) $\delta_E(\cdot)$ is continuous on $[0, 2)$;*

*(ii) $\delta_E(\cdot)$ is strictly increasing on $[\varepsilon_0, 2]$;*

*(iii) If, in addition, $E$ is uniformly convex, $r > 0$ and $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$, then*

$$\|\lambda x + (1 - \lambda)y\| \leq r\left[1 - 2\min\{\lambda, 1 - \lambda\}\delta_E\left(\frac{\|x - y\|}{r}\right)\right] \ \text{for all } \lambda \in [0, 1]. \quad (2.4)$$

*In particular, taking $\lambda = \frac{1}{2}$,*

$$\left\|\frac{x + y}{2}\right\| \leq r\left[1 - \delta_E\left(\frac{\|x - y\|}{r}\right)\right]. \quad (2.5)$$

*(iv) Each uniformly convex Banach space has the Kadec-Klee property, i.e.,*

$$\text{weak-} \lim_{n \to \infty} x_n = x \ \text{ and } \ \lim_{n \to \infty} \|x_n\| = \|x\| \ \text{ implies } \ \lim_{n \to \infty} x_n = x.$$

The following conclusion is well known. For the more detail, see the references [5, 9, 27].

**Lemma 2.4.** *Let $C$ be a nonempty closed convex subset of a reflexive Banach space $E$. Assume that $f : C \to (-\infty, +\infty)$ is a proper convex lower semi-continuous and coercive function (i.e., $\lim_{\|x\| \to \infty} f(x) = \infty$). Then there exists $x \in C$ such that*

$$f(x) = \inf_{y \in C} f(y).$$

# 3 Pazy type Fixed Point Theorems with the partial order

Let $K$ be a nonempty closed convex subset of an ordered Banach space $(E, \leq)$ and let $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping. In this section, we consider the Picard iteration defined by

$$x_{n+1} = Tx_n = T^n x_0 \tag{3.1}$$

for fixed $x_0 \in K$ and each positive integer $n \geq 1$.

**Definition 3.1.** *Let $T$ be a mapping on a Banach space $E$ and $x_0 \in D(T)$, the domain of $T$. Then the set of points, denoted by $O(x_0)$ is called the* orbit *of $x_0$ under $T$ if*

$$O(x_0) = \{T^n x_0; n = 0, 1, 2, \cdots\},$$

*and its closure is called the* closed orbit.

The following technical lemma plays a key role in the proof of main results of this work, which may be found in the reference [25]. For the completeness, we give its proof.

**Lemma 3.1.** *Let $(E, \|\cdot\|)$ be a Banach space with the partial order "$\leq$". Assume that a sequence $\{x_n\} \subset E$ is monotonic.*

*(i) If there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k}$ weakly converges to some point $x \in E$, then $x_n \leq x$ for all positive integer $n$ when $\{x_n\}$ is increasing or $x_n \geq x$ for all positive integer $n$ when $\{x_n\}$ is decreasing.*

*(ii) If $(E, \|\cdot\|)$ is reflexive and $\{x_n\}$ is bounded with respect to the norm $\|\cdot\|$, then $x_n$ weakly converges to some point $x \in E$.*

*Proof.* (i) Assume that $\{x_n\}$ is increasing, i.e., $x_n \leq x_{n+1}$ for all positive integer $n$. Then for given $n$, there exists a positive integer $k_0$ such that $n < n_{k_0}$, and hence, $x_n \leq x_{n_{k_0}} \leq x_{n_k}$ for all $k > k_0$. It follows from Lemma 2.1 (i) and the weak convergence of $x_{n_k}$ that $x_n \leq x$. Since $n$ is arbitrary, $x_n \leq x$ for all positive integer $n$.

Using the same proof technique, it is easy to prove that $x_n \geq x$ for all positive integer $n$ when $\{x_n\}$ is decreasing.

(ii) Without loss of generality, we may assume that $\{x_n\}$ is increasing. It follows from the reflexivity of $E$ and the boundedness of $\{x_n\}$ that $\{x_n\}$ is weakly sequentially compact. Then we may choose two subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ in $\{x_n\}$ such that $x_{n_k}$ and $\{x_{n_j}\}$ weakly converge to $x$ and $y$, respectively. We must have $x = y$. In fact, using the the same proof technique as (i), for fixed $n_k$, there is large enough $n_j$ such that $x_{n_k} \leq x_{n_j}$. Then by Lemma 2.1 (i), we have $x_{n_k} \leq y$, and so for all $k$, $x_{n_k} \leq y$. Again using Lemma 2.1 (i), we have $x \leq y$. Using the same proof technique, we also have $y \leq x$. Thus $x = y$. Which means that any subsequences $\{x_{n_k}\}$ of $\{x_n\}$ weakly converges to $x$, and hence, $\{x_n\}$ weakly converges to $x$. $\qquad\square$

Now we prove some existence theorems of fixed points with the partial order "$\leq$" of monotone $\alpha$-nonexpansive mappings in a partial order Banach space.

**Theorem 3.2.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ with the partial order "$\leq$" and $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping.*

(i) *If $x_0 \leq Tx_0$ and the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$, then $F_{\geq}(T) \neq \emptyset$.*

(ii) *If $F_{\geq}(T) \neq \emptyset$, then there exists $x_0 \in K$ such that the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$.*

*Proof.* (i) Since $x_0 \leq Tx_0$, it follows from the monotonicity of $T$ that $Tx_0 \leq T^2x_0$, and so

$$T^2x_0 \leq T^3x_0, \cdots, T^nx_0 \leq T^{n+1}x_0, \cdots.$$

Namely,

$$x_0 \leq Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \cdots.$$

By Lemma 3.1, there exists $x \in E$ such that $T^nx_0$ weakly converges to $x$ and $T^nx_0 \leq x$ for all positive integer $n$. Since the closed convexity of $K$ implies that $K$ is weakly sequentially closed, we have $x \in K$.

Let $K_n = \{y \in K; T^nx_0 \leq y\}$ for all positive integer $n$. Clearly, for each $n$, $K_n$ is closed convex and $x \in K_n$ for all positive integer $n$. Let $C = \bigcap_{n=1}^{\infty} K_n$. Clearly, $C \subset K$ is a nonempty

closed convex ($x \in C$). Since the orbit $O(x_0) = \{T^n x_0; n = 0, 1, 2, \cdots\}$ is bounded with respect to the norm $\|\cdot\|$, we may define a function $f : C \to [0, +\infty)$ as follows

$$f(y) = \limsup_{n\to\infty} \|T^n x_0 - y\| \text{ for all } y \in C.$$

It is obvious that $f$ is a proper convex, continuous and coercive function. It follows from Lemma 2.4 that there exists $z \in C$ such that

$$f(z) = \limsup_{n\to\infty} \|T^n x_0 - z\| = \inf_{y\in C} f(y) = r. \tag{3.2}$$

By the definition of $C$, we have

$$T^n x_0 \leq z \text{ for all positive integer } n,$$

and hence,

$$T^n x_0 \leq T^{n+1} x_0 \leq Tz \text{ for all positive integer } n,$$

which implies that $Tz \in C$. Thus $\dfrac{z + Tz}{2} \in C$, and so, it follows from (3.2) that

$$r = f(z) \leq f\left(\frac{z + Tz}{2}\right) \quad \text{and} \quad r = f(z) \leq f(Tz). \tag{3.3}$$

On the other hand, it follows from the definition of $T$ that

$$\|T^{n+1} x_0 - Tz\|^2 \leq \alpha \|T^{n+1} x_0 - z\|^2 + \alpha \|Tz - T^n x_0\|^2 + (1 - 2\alpha)\|T^n x_0 - z\|^2,$$

and so,

$$(\limsup_{n\to\infty} \|T^{n+1} x_0 - Tz\|)^2 \leq \alpha(\limsup_{n\to\infty} \|T^{n+1} x_0 - z\|)^2 + \alpha(\limsup_{n\to\infty} \|Tz - T^n x_0\|)^2$$
$$+ (1 - 2\alpha)(\limsup_{n\to\infty} \|T^n x_0 - z\|)^2,$$

that is,

$$(1 - \alpha)(\limsup_{n\to\infty} \|Tz - T^n x_0\|)^2 \leq (1 - \alpha)(\limsup_{n\to\infty} \|T^n x_0 - z\|)^2.$$

Thus, we have

$$f(Tz) = \limsup_{n\to\infty} \|T^n x_0 - Tz\| \leq \limsup_{n\to\infty} \|T^n x_0 - z\| = f(z) = r. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$r = f(z) = f(Tz) \geq 0.$$

We claim $z = Tz$. In fact, suppose $r = 0$. Then

$$\lim_{n\to\infty} \|T^n x_0 - Tz\| = \lim_{n\to\infty} \|T^n x_0 - z\| = 0,$$

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which means $z = Tz$. Next suppose $r > 0$. It follows from the definition of the upper limit "lim sup" that for all $\varepsilon > 0$, there exists a positive integer $N$ such that

$$\|T^n x_0 - Tz\| < r + \varepsilon \text{ and } \|T^n x_0 - z\| < r + \varepsilon \text{ for all positive integer } n > N.$$

It follows from Lemma 2.3 that for all positive integer $n > N$,

$$\left\|T^n x_0 - \frac{z + Tz}{2}\right\| = \left\|\frac{(T^n x_0 - z) + (T^n x_0 - Tz)}{2}\right\| \leq (r + \varepsilon)\left[1 - \delta_E\left(\frac{\|z - Tz\|}{r + \varepsilon}\right)\right].$$

Without loss of generality, we may assume that $\varepsilon < 1$. Then the above inequality can be rewritten as follow:

$$\left\|T^n x_0 - \frac{z + Tz}{2}\right\| \leq (r + \varepsilon)\left[1 - \delta_E\left(\frac{\|z - Tz\|}{r + 1}\right)\right],$$

and so, we have

$$f\left(\frac{z + Tz}{2}\right) = \limsup_{n \to \infty}\left\|T^n x_0 - \frac{z + Tz}{2}\right\| \leq (r + \varepsilon)\left[1 - \delta_E\left(\frac{\|z - Tz\|}{r + 1}\right)\right].$$

From (3.3), it follows that

$$r\delta_E\left(\frac{\|z - Tz\|}{r + 1}\right) \leq (r + \varepsilon)\delta_E\left(\frac{\|z - Tz\|}{r + 1}\right) \leq r - f\left(\frac{z + Tz}{2}\right) + \varepsilon \leq \varepsilon.$$

Since $\varepsilon$ is arbitrary, we have

$$r\delta_E\left(\frac{\|z - Tz\|}{r + 1}\right) = 0,$$

and hence, $z = Tz$ by Lemma 2.3 (ii). Clearly, $z \geq x_0$, and so $z \in F_\geq(T)$.

(ii) Let $z \in F_\geq(T)$. Then there exists a $x_0 \in K$ such that $x_0 \leq z$. It follows from the monotonicity of $T$ that $Tx_0 \leq Tz = z$, and hence, $T^2 x_0 = T(Tx_0) \leq Tz = z$. By such analogy, we have

$$T^n x_0 \leq Tz = z \text{ for all positive integer } n.$$

From Lemma 2.2, it follows that

$$\|T^n x_0 - z\| \leq \|x_0 - z\| \text{ for all positive integer } n,$$

which implies the boundedness of the sequence $\{T^n x_0\}$. This completes the proof. $\quad\square$

Using the same proof technique of Theorem 3.2 (the only change is $K_n = \{y \in K : y \leq T^n x_0\}$ and $T^{n+1} x_0 \leq T^n x_0$), the following theorem is easy to be proved.

**Theorem 3.3.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ with the partial order "$\leq$" and let $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping.*

(i) If $x_0 \geq Tx_0$ and the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$, then $F_{\leq}(T) \neq \emptyset$.

(ii) If $F_{\leq}(T) \neq \emptyset$, then there exists $x_0 \in K$ such that the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$.

**Theorem 3.4.** *Let $(E, \|\cdot\|)$ be a uniformly convex ordered Banach space with the partial order "$\leq$" with respect to closed convex cone $P$ and let $T : P \to P$ be a monotone $\alpha$-nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if the orbit $O(0)$ is bounded with respect to the norm $\|\cdot\|$.*

*Proof.* Since $T(P) \subset P$, for all $x \in F(T)$, we have $x \geq 0$, and so $x \in F_{\geq}(T)$. Then $F(T) \subset F_{\geq}(T) \subset F(T)$, and so, $F(T) = F_{\geq}(T)$.

"Sufficiency". It follows from the definition of the partial order "$\leq$" that $x_0 = 0 \leq T0 = Tx_0$. Then the conclusion directly follows from Theorem 3.2.

"Necessity". Take $x \in F(T) = F_{\geq}(T)$. Then $x \geq 0$, and hence,

$$x = Tx \geq T0, x = Tx \geq T^2 0, \cdots, x = Tx \geq T^n 0, \cdots.$$

That is, $x \geq T^n 0$ for all positive integer $n$. It follows from Lemma 2.2 that

$$\|T^{n+1} 0 - x\| \leq \|T^n 0 - x\| \leq \cdots \leq \|0 - x\| = \|x\|.$$

This means that the orbit $O(0)$ is bounded with respect to the norm $\|\cdot\|$. $\qquad \square$

# 4 Weak and strong convergence of Picard iteration

**Definition 4.1.** *Let $(E, \|\cdot\|)$ be a Banach space with the partial order "$\leq$". The norm $\|\cdot\|$ is called* monotonic *if*

$$\|x\| \leq \|y\| \text{ for all } x, y \in E \text{ with } 0 \leq x \leq y.$$

It is well known that if the cone $P$ is normal, by Lemma 2.1, if and only if there exists a equivalent norm $\|\cdot\|_1$ which is monotonic.

**Theorem 4.1.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ with the partial order "$\leq$" with respect to the normal cone $P$ and let $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping. Assume that $x_0 \leq Tx_0$ and the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$. If the norm $\|\cdot\|$ is monotonic, then $T^n x_0$ weakly converges to some point $z \in F_{\geq}(T)$.*

*Proof.* Using the same proof technique of Theorem 3.2, we obtain the following:

(i) $T^n x_0$ weakly converges to some point $x \in K$ and $T^n x_0 \leq x$ for all positive integer $n$. Furthermore,

$$x \in C = \bigcap_{n=1}^{\infty} \{y \in K; T^n x_0 \leq y\}.$$

(ii) Let $f(y) = \limsup_{n \to \infty} \|T^n x_0 - y\|$ for all $y \in C$. Then there exists $z \in C$ such that $z = Tz \geq T^n x_0$ for all positive integer $n$ and

$$f(z) = \inf\{f(y); y \in C\} \leq f(x).$$

Next we prove $z = x$. It follows from Lemma 2.1 (i) that

$$T^n x_0 \leq x \leq z = Tz.$$

Thus, we have

$$0 \leq x - T^n x_0 \leq z - T^n x_0,$$

which implies that

$$\|x - T^n x_0\| \leq \|z - T^n x_0\|$$

by the monotonicity of the norm $\|\cdot\|$. Consequently,

$$f(x) = \limsup_{n \to \infty} \|x - T^n x_0\| \leq \limsup_{n \to \infty} \|z - T^n x_0\| = f(z) \leq f(x),$$

and hence, $f(x) = f(z) = r$. Using the same proof technique of Theorem 3.2, we have $x = z$ also. This completes the proof. $\qquad\square$

Similarly, we also the following theorem.

**Theorem 4.2.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ with the partial order "$\leq$" and let $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping. Assume that $x_0 \geq Tx_0$ and the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$. If the norm $\|\cdot\|$ is monotonic, then $T^n x_0$ weakly converges to some point $z \in F_{\geq}(T)$.*

Let the intersection of the domain $D(T)$ of monotone $\alpha$-nonexpansive mapping $T$ and the cone $P$ is nonempty, i.e., $D(T) \cap P \neq \emptyset$. The following strongly convergent theorems may be obtained.

**Theorem 4.3.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ with the partial order "$\leq$" and let $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping. Assume that $0 \leq x_0 \leq Tx_0$ and the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$. If the norm $\|\cdot\|$ is monotonic, then $T^n x_0$ strongly converges to some point $z \in F_{\geq}(T)$.*

*Proof.* It follows from Theorem 4.1 that there exists $z \in F_{\geq}(T)$ such that $T^n x_0$ weakly converges to some point $z \in K$ and

$$0 \leq x_0 \leq T^n x_0 \leq T^{n+1} x_0 \leq z \text{ for all positive integer } n. \quad (4.1)$$

Then by the monotonicity of the norm, we have

$$0 \leq \|x_0\| \leq \|T^n x_0\| \leq \|T^{n+1} x_0\| \leq \|z\| \text{ for all positive integer } n. \quad (4.2)$$

That is, the sequence $\{\|T^n x_0\|_1\}$ is monotone increasing and bounbed. So there is a real number $\gamma$ such that

$$\lim_{n \to \infty} \|T^n x_0\| = \gamma \leq \|z\|. \quad (4.3)$$

From the weakly lower semi-continuity of the norm, it follows that

$$\|z\|_1 \leq \liminf_{n \to \infty} \|T^n x_0\| = \lim_{n \to \infty} \|T^n x_0\| = \gamma \leq \|z\|,$$

and hence,

$$\lim_{n \to \infty} \|T^n x_0\| = \|z\|.$$

By Lemma 2.3 (iv), we have $\lim_{n \to \infty} T^n x_0 = z$. This completes the proof. $\quad \square$

**Theorem 4.4.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ with the partial order "$\leq$" and let $T : K \to K$ be a monotone $\alpha$-nonexpansive mapping. Assume that $0 \geq x_0 \geq T x_0$ and the orbit $O(x_0)$ is bounded with respect to the norm $\|\cdot\|$. If the norm $\|\cdot\|$ is monotonic, then $T^n x_0$ strongly converges to some point $z \in F_{\leq}(T)$.*

*Proof.* It follows from Theorem 4.2 that there exists $z \in F_{\leq}(T)$ such that $T^n x_0$ weakly converges to $z$ and

$$0 \geq x_0 \geq T^n x_0 \geq T^{n+1} x_0 \geq z \text{ for all positive integer } n.$$

Then

$$0 \leq -x_0 \leq -T^n x_0 \leq -T^{n+1} x_0 \leq -z \text{ for all positive integer } n,$$

and so,

$$0 \leq \|x_0\| \leq \|T^n x_0\| \leq \|T^{n+1} x_0\| \leq \|z\| \text{ for all positive integer } n.$$

The remainder of the proof are the same as ones of Theorem 4.3, we omit it. $\quad \square$

Combining Theroem 3.4 and 4.1, the following conclusion is easy to be showed.

**Corollary 4.5.** *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space with the partial order "$\leq$" with respect to the closed convex cone $P$ and let $T : P \to P$ be a monotone $\alpha$-nonexpansive mapping with $F(T) \neq \emptyset$. If the norm $\|\cdot\|$ is monotonic, then $T^n 0$ strongly converges to some point $z \in F(T)$.*

**Corollary 4.6.** *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space with the partial order "$\leq$" with respect to the closed convex $P$ and let $T : P \to P$ be a monotone nonexpansive mapping with $F(T) \neq \emptyset$. If the norm $\|\cdot\|$ is monotonic, then for all $x \in P$ with $x \leq Tx$, $T^n x$ strongly converges to some point $z \in F(T)$.*

*Proof.* It follows from Theroem 3.4 that the orbit $O(0)$ is bounded with respect to the norm $\|\cdot\|$. Then by the monotonicity of $T$, for all $x \in P$ (i.e.,$0 \leq x$), we have $T0 \leq Tx$, $T^2 0 \leq T^2 x$, and so by the same manner, we obtain

$$T^n 0 \leq T^n x \text{ for all positive integer } n.$$

Since $T$ be a monotone nonexpansive mapping, we have

$$\|T^n 0 - T^n x\| \leq \|T^{n-1} x - T^{n-1} 0\| \leq \cdots \leq \|x - 0\| = \|x\| \text{ for all positive integer } n.$$

So the orbit $O(x)$ is bounded with respect to the norm $\|\cdot\|$. Then the conclusion directly follows from Theorem 4.3. $\qquad\square$

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